Lemma. If f is uniformly continuous on a set A, and if $B \subseteq A$, then f is also uniformly continuous on B.

Pf. Suppose f is unif. cont. on A, so $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x, c \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f is restricted to B, and $x, c \in B$ and $|x - c| < \delta$, also $x, c \in A$, so the above applies and $|f(x) - f(c)| < \varepsilon$. $\blacksquare$

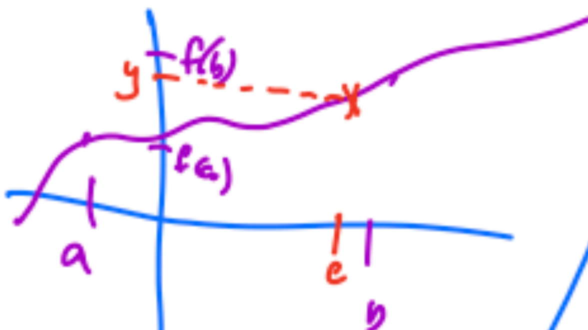
Another amazing fact about continuous functions.

Thm (The continuous image of a connected set is connected). That is, If $f: A \to \mathbb{R}$ is continuous and A is connected, then $f(A)$ is connected.

Pf — (topological).

Corollary: **Intermediate Value Theorem.**

If $f:[a,b] \to \mathbb{R}$ is continuous, and if $y \in \mathbb{R}$ s.t. $f(a) < y < f(b)$ or $f(b) < y < f(a)$, Then there exists $c \in (a,b)$ such that $f(c) = y$.



direct result of $[a,b]$ connected, f cont. $\Rightarrow$ the interval $[f(a), f(b)]$ or $[f(b), f(a)]$ must be inside the image $= f([a,b])$

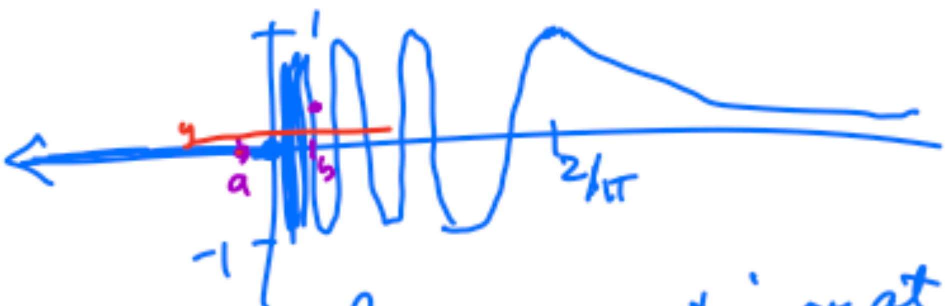
A Property of functions that are not necessarily continuous is called the Intermediate Value Property: We say a function $g:[a,b] \to \mathbb{R}$ satisfies the Intermediate Value Property if whenever $g(x) < g(y)$ for some $x, y \in [a,b]$ and $g(x) < p < g(y)$, there $\exists$ q between x & y

s.t. $g(q) = p$.

Shocking: There are functions that have this property that are not continuous.

Example: Let $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{for } x > 0 \\ 0 & \text{for } x \le 0. \end{cases}$

$f : \mathbb{R} \to \mathbb{R}$



f is not continuous at 0.
(we showed this before)

It turns out that even if $a \le 0$ and $b > 0$, and $f(a) < f(b)$ or $f(a) > f(b)$ and y is between $f(a)$ & $f(b)$, then $\exists$ an infinite # of points q s.t. $f(q) = y$.

Derivatives

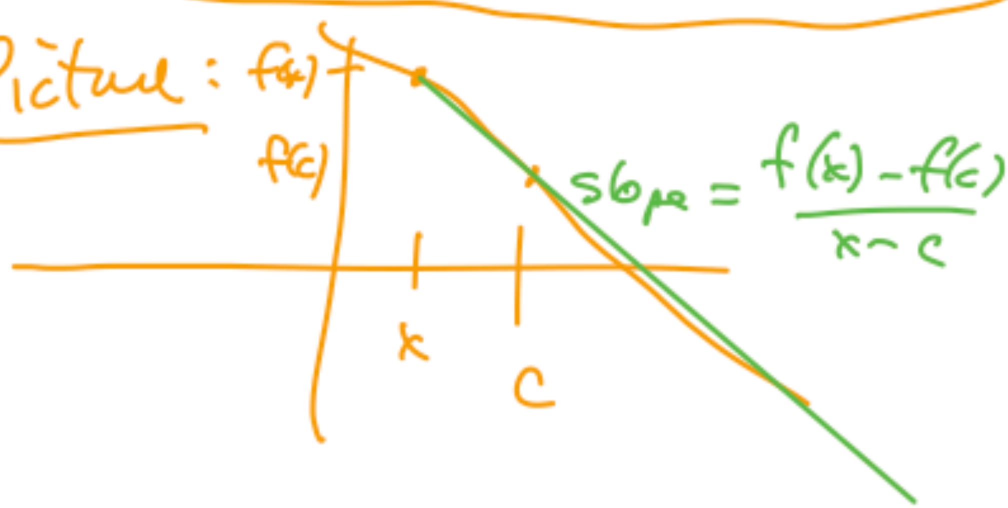
Let f be a function defined on an interval I, $f: I \to \mathbb{R}$.
For $c \in I$, we say that f is <u>differentiable</u> at c if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.
(& we call the limit $\frac{df}{dx}(c)$ or $f'(c)$, the derivative at c.)

FYI, change variables via $x = c+h$ for $h \in \mathbb{R}$,

$$\lim_{c+h \to c} \frac{f(c+h)-f(c)}{c+h-c} = f'(c)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c).$$

Picture:



We say f is differentiable on I (an interval or set) if f is differentiable at every point of I.

Thm (differentiable $\Rightarrow$ continuous)

If $f: I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at $c \in I$.

Pf. Sp. $f: I \to \mathbb{R}$ is diff'ble at $c \in I$. Then $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists. Then

$$\lim_{x \to c} f(x) = \lim_{x \to c} f(c) + (x-c) \cdot \frac{f(x) - f(c)}{(x-c)}$$

$$\underset{ALT}{=} f(c) + 0 \, f'(c) = f(c). \quad \blacksquare$$

Note $c \in I$ is not isolated, so f is continuous at $c \Leftrightarrow \lim_{x \to c} f(x) = f(c)$.

Algebraic Derivative Thm.

Suppose f, g are function on an interval I and they are differentiable at $c \in I$. Let $a \in \mathbb{R}$.

Then:

ⓐ $(f \pm g)'(c) = f'(c) \pm g'(c)$

ⓑ $(af)'(c) = a f'(c)$.

ⓒ $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

ⓓ $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$, if $g(c) \neq 0$.

ⓔ $(\sqrt{f})'(c) = \frac{f'(c)}{2\sqrt{f(c)}}$, if $f(c) > 0$.

Thm: Chain Rule.

If $f: I \to \mathbb{R}$ is differentiable at c, and if $g: J \to \mathbb{R}$ is diff'ble at $f(c) \in J$, then $g \circ f$ is diff'ble at c, and $(g \circ f)'(c) = g'(f(c)) f'(c)$.

Example Proof: Product rule

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c)$$

Given: f, g are diff'ble at $c \in I$.

Then

$$\lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}$$

$$= \lim_{x\to c} f(x)\frac{g(x)-g(c)}{x-c} + \frac{f(x)-f(c)}{x-c}g(c)$$

Note: $\lim_{x\to c} f(x) = f(c)$ is f is diff'ble at c $\Rightarrow$ f is continuous at c

$$\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c) \quad \text{(given)}$$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) \quad \text{(given)}.$$

By ALT

$$= f(c)g'(c) + f'(c)g(c). \quad \boxtimes$$

Interesting Fact.

Darboux Theorem. If f is differentiable on an interval I, then f' satisfies the intermediate value property.

(i.e. derivatives of fcns are *almost* continuous)

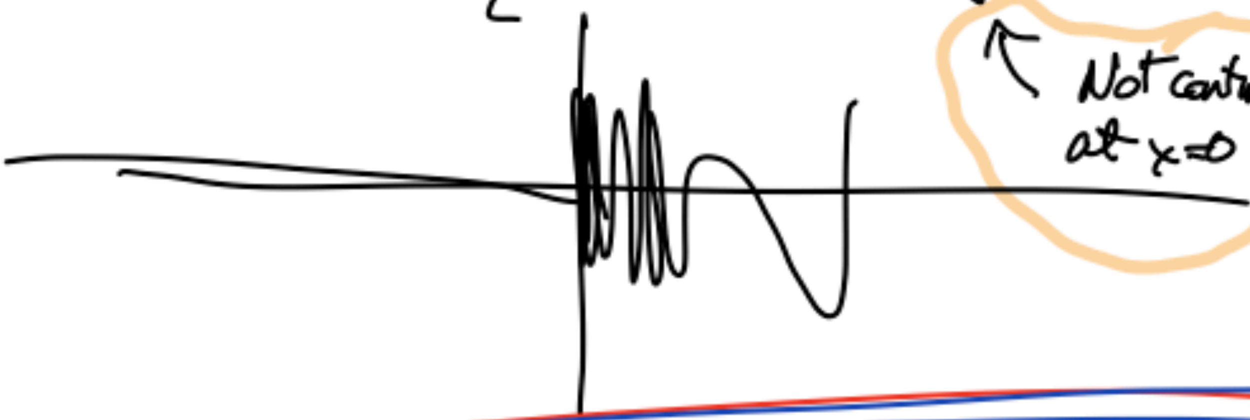
Example: Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

It turns out:

(a) f is diff'ble everywhere, including 0.

(b) $f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$

Not continuous at $x=0$



<u>Thm</u> (Interior extremum theorem)
Suppose $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and continuous on $[a,b]$. Suppose that f achieves its maximum (or

minimum) at $c \in (a,b)$, so that $f(c) \geq f(x) \quad \forall x \in [a,b]$.

Then $f'(c) = 0$. ($\leq$ for min case)

<u>Pf.</u> With given,

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c).$$

Let (x_n) be a sequence of points in (a,b) approaching c from the right side (eg $x_n = \frac{1}{n} + c$). Let (y_n) be a sequence of pts in (a,b) approaching c from the right.

Then
$$f'(c) = \lim_{n \to \infty} \frac{f(x_n)-f(c)}{x_n - c}$$

$$f(c) \geq f(x_n) \Rightarrow f(x_n) - f(c) \leq 0$$

$$x_n > c \Rightarrow x_n - c > 0$$

$$\Rightarrow \frac{f(x_n)-f(c)}{x_n - c} \leq 0 \quad \forall n$$

$$\Rightarrow f'(c) \leq 0 \text{ by the OLT}$$

(*)

Similarly $f(c) \geq f(y_n)$

$y_n < c$

$$\Rightarrow \frac{f(y_n) - f(c)}{y_n - c} \geq 0$$

$\Rightarrow f'(c) = \lim$ " ≥ 0 by OLT. $(**)$

By $(*)$ & $(**)$, $f'(c) = 0$. $\boxed{/}$

<u>Rolle's Thm</u>. Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and diff'ble on (a,b). If $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

<u>Proof</u>. With given, either

① f is constant on $[a,b]$.

② $\exists\, c \in (a,b)$ s.t. $f(c) > f(a) = f(b)$

③ $\exists\, c_2 \in (a,b)$ s.t. $f(c_2) < f(a) = f(b)$

By the Extreme Value Theorem, the min & max of f are achieved on $[a,b]$, since $[a,b]$ is compact. If case ② or ③ occurs, there must be an interior max or min $x \Longrightarrow f'(x) = 0$. If case ①, $f'(x) = 0 \;\forall x \in [a,b]$. ∎